PMM U.S.S.R.,Vol.50,No.3,pp.350-357,1986
Printed in Great Britain

NON-STATIONARY DYNAMIC CONTACT PROBLEM FOR A PERIODIC SYSTEM OF STAMPS UNDER ARBITRARY LOADING*

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The contact stresses are determined under a periodic system of stamps on the boundary of a homogeneous plastic half-plane and moving under the action of a load that is arbitrary in time. Unlike /1/, a more general problem is solved when the system of stamps is firstly invariant not only with respect to shear (translations) along the half-plane boundary by vectors that are multiples of a certain vector $\mathbf{a}_0$, but also with respect to reflections in a periodic system of parallel planes π_k $(k = 0, \pm 1, \pm 2, \ldots)$ perpendicular to the vector $\mathbf{a}_0$ and separated from each other by a distance $l = |\mathbf{a}_0|/2$ (these transformations form a group C_{1h} /2/, and secondly, the load is not assumed to be identical for all stamps (Fig.1).

1. Let us consider a set X of points on the Ox axis invariant under transformations of the group C_{1h} (Fig.2a). We will call the segment $[0, 2l]$ basic and the segment $[0, l]$ of the axis the basic unit cell (UC) and ascribe the symbols X_0 and X_{01} to them, respectively. The UC system

$$X_{mn} = g_{m1}h_nX_{01} \quad (m = 0, \pm 1, \pm 2, \ldots; \quad h_n \in H; \; n = 1, 2)$$

forms the covering of the set X. A translation by the vector ma_0 is denoted by g_{m1}, and H is a point group consisting of two elements, the identity transformation h_1 and the reflection h_2 in the plane Π_0.

We introduce local coordinate systems on each of the UC that go over into each other under symmetry transformations (Fig.2a). We will later call invariant the system formed by these local coordinate systems. If a point x of the half-plane boundary is on the UC X_{mn}, we shall denote it and its abscissa in the invariant coordinate system by x_{mn}.

We introduce the functions $\psi_n(x_0)$ $(n = 1, 2)$, defined on the basic segment X_0, as follows: $\psi_n(x_0) = \delta_{pn}$ $(x_0 \in X_{0p}$; δ_{np} is the Kronecker delta and $n, p = 1, 2)$. We decompose each of these functions into components, being transformed into non-equivalent irreducible representations of the point group $H_\mathbf{k}$ of a certain vector $\mathbf{k} \in \Omega$ (Ω is the Brillouin zone) /2, 3/

$$\psi_n(x_0) = \sum_{\mu=1}^{M_\mathbf{k}} \psi_{n\mu}(x_0, \mathbf{k}), \quad \psi_{n\mu}(x_0, \mathbf{k}) = \frac{1_\mu}{N_k} \sum_{h \in H_\mathbf{k}} \tau^\mu(h^{-1})\, h\psi_n(x_0)$$

In these formulas $M_\mathbf{k}$ is the number of non-equivalent irreducible representations of the group $H_\mathbf{k}$, τ^μ is a matrix element of this representation (this matrix is one dimensional for the group C_{1h}), and $N_\mathbf{k}$ is the order of the group $H_\mathbf{k}$.

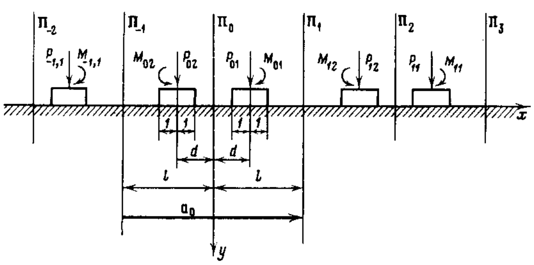

Fig.1

It follows from the theory of group representations that the functions

$$\Phi_{n\mu j}(x, \mathbf{k}) = h_j'\psi_{n\mu}(x_0, h_j'\mathbf{k}) \exp(-ih_j'\mathbf{k} \cdot m\mathbf{a}_0) \tag{1.1}$$
$$(j = 1, 2, \ldots, L; \; x = g_{m1}x_0; \; m = 0, \pm 1, +2, \ldots)$$

*Prikl.Matem.Mekhan.,50,3,461-469,1986

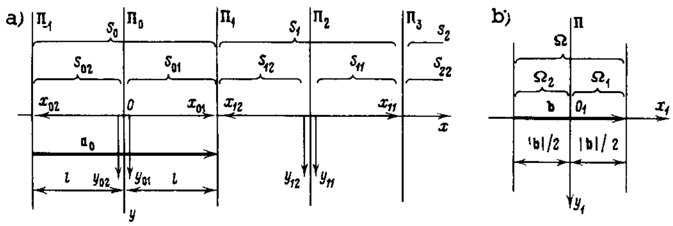

Fig.2

for fixed $n, \mathbf{k}$ and μ are transformed into an irreducible representation of the group C_{1h}.

The h_j' $(j = 1, 2, \ldots, L)$ in (1.1) denote elements of the group H that take part in its decomposition into left affine classes in the subgroup $H_{\mathbf{k}}$.

For a bounded function $f(x)$ on X the series

$$F(x_0, \alpha) = \sum_{m=-\infty}^{\infty} f(g_{m1}x_0)\exp(im\alpha), \quad x_0 \in X_0 \tag{1.2}$$

defines a generalized function on the segment $\Gamma\{-\pi \leqslant \alpha \leqslant \pi\}$ with identical ends.

Then

$$f(x) = \frac{1}{2\pi}(F, \exp(-im\alpha))$$

This equation can be written thus

$$f(x) = \frac{1}{|\mathbf{b}|}(F, \exp(-i\mathbf{k}\cdot m\mathbf{a}_0))$$

The function $F(x_0, \mathbf{k}) = F(x_0, \alpha)$ is considered generalized in the Brillouin zone Ω, hence $\mathbf{b} = 2\pi\mathbf{a}_0/|\mathbf{a}_0|^2$ and $\mathbf{k} = \alpha\mathbf{b}/(2\pi)$.

Let Δ_X denote a set of bounded functions on X for which the discrete Fourier transform (1.2) has the following form:

$$F(x_0, \mathbf{k}) = F_0(x_0, \mathbf{k}) + \sum_{j=1}^{J} F_1(x_0, \mathbf{k}_j)\,\delta(\mathbf{k} - \mathbf{k}_j) \tag{1.3}$$

Here $F_0(x_0, \mathbf{k})$ is a piecewise continuous function defined in $\mathbf{k}$ everywhere in Ω with the exception of a finite number of points, and is integrable in Ω in the sense of the Cauchy principal value, while $\delta(\mathbf{k})$ is the delta function.

By using the theory of group representations the following assertion can be proved: an arbitrary function of Δ_X has a discretely continual decomposition

$$f(x) = \frac{1}{|\mathbf{b}|}\int_{\Omega_1}\sum_{n,j=1}^{2} F_0(h_jx_{0n}, h_j\mathbf{k})\,\Phi_{n1j}(x, \mathbf{k})\,d\Omega_{\mathbf{k}} + \tag{1.4}$$
$$\frac{1}{|\mathbf{b}|}\sum_{j=1}^{J}\sum_{\mu=1}^{M_j}\sum_{n=1}^{2} F_1(x_{0n}, \mathbf{k}_j)\,\Phi_{n\mu1}(x, \mathbf{k}_j)$$

in the functions $\Phi_{n\mu j}(x, \mathbf{k})$ transformed in irreducible representations of the group C_{1h}. Here M_j is the number of non-equivalent irreducible representations of the group of the vector $\mathbf{k}_j$, and Ω_1 is the unit cell (UC) of the Brillouin zone, i.e., the segment $[0, |b|/2]$ of the Ox_1 axis (Fig.2b).

Note that the algorithm elucidated above permits decomposition of a function of more general form on X into generalized symmetric components (i.e., transformation in irreducible representations of the group C_{1h}) than does the analogous algorithm in /4/.

2. A system of stamps invariant with respect to transformations of the group C_{1h} (Fig.1) is located on the boundary of a homogeneous elastic half-plane. Numbers are ascribed to the stamps in conformity with the numbering of the UC on which they are located. We assume the varaibles x and t to be dimensionless, we take the length scale equal to a (a is half the width of the stamp), and the time to a/c_2 (c_2 is the velocity of transverse elastic wave propagation). Friction between the stamps and the half-plane is assumed non-existent and the length of the contact area is assumed to equal the width of the stamp.

Lumped vertical forces $P_{mn}(t)$ and moments $M_{mn}(t)$ $(m = 0, \pm1, \pm2, \ldots, n = 1, 2; t > 0)$ act on the stamps. It is assumed that all the stamps and the half-plane are at rest at $t \leqslant 0$. For fixed t the forces and moments applied to the stamps can be considered to be functions given in the set X with symmetry group C_{1h} having at one point on each of the UC

$$P_{mn}(t) = P(x, t), \; M_{mn}(t) = M(x, t) \; (x \in X_{mn}; \; m = 0, \pm 1, \pm 2, \ldots; \; n = 1, 2)$$

We assume that these are functions of Δ_X, which means that the decomposition (1.4) holds for them. There results from the linearity of the formulation of the problem and the symmetry of the system of stamps /5/ that the same decomposition also holds in the invariant coordinate system for both the contact stresses $p(x, t)$ and displacements $w(x, t)$ of points of the stamp bases, and moreover, the solution of the problem for arbitrary loading reduces to the solution of a series of generalized symmetric problems in each of which the load is transformed into an irreducible representation of the group C_{1h}.

We note that because of the boundedness of the excitation propagation velocity in an elastic medium, the stresses under the stamp during a finite time interval $0 \leqslant t \leqslant t_0$ will depend on the motion of this stamp and of some finite number of neighbours. Hence, the stresses under the stamp can be determined by solving the problem of a half-plane with a system of a finite number of stamps dependent on t_0. However, these same stresses can be found by considering an infinite system of stamps, which is an extension of the preceding, with an arbitrary load from Δ_X applied to the appended stamps. In the last case, the application of the theory of group representations is possible, which results in decay of the problem into a series of generalized symmetric problems in each of which equations must be solved for just two stamps (and in some cases even for one). Such an extension should naturally be performed so that the extended system has the greatest symmetric properties.

We will consider the case when the irreducible representation to which this generalized symmetric problem corresponds is two-dimensional $(0 < \alpha < \pi)$.

By using the transformation

$$\Phi_{n1j}(x, \mathbf{k}) = \frac{1}{2}[\theta_{11}(x, \mathbf{k}) + (-1)^{n+j}\theta_{22}(x, \mathbf{k})] - (-1)^j \frac{i}{2}[\theta_{21}(x, \mathbf{k}) - (-1)^{n+j}\theta_{12}(x, \mathbf{k})] \quad (n, j = 1, 2) \tag{2.1}$$

we transfer from the functions $\Phi_{n1j}(x, \mathbf{k})$ $(n, j = 1, 2)$ to the functions $\theta_{jn}(x, \mathbf{k})$ $(j, n = 1, 2)$ that are transformed for fixed n into a two-dimensional real irreducible representation of the group

$$\theta_{jn}(gx, \mathbf{k}) = \sum_{q=1}^{2} \tau_{jq}^{\mathbf{k}}(g)\theta_{qn}(x, \mathbf{k}) \quad (g \in C_{1h}; \; j, n = 1, 2) \tag{2.2}$$

$$\tau_{jj}^{\mathbf{k}}(g_{m1}) = \cos m\alpha, \quad \tau_{jq}^{\mathbf{k}}(g_{m1}) = (-1)^q \sin m\alpha \quad (j \neq q)$$

$$\tau_{jj}^{\mathbf{k}}(g_{m2}) = (-1)^{j-1}\cos m\alpha, \quad \tau_{jq}^{\mathbf{k}}(g_{m2}) = -\sin m\alpha \quad (j \neq q)$$

$$(j, q = 1, 2)$$

Here $\alpha = \mathbf{k}\cdot\mathbf{a}_0$, g_{m1} are translations by the vector $m\mathbf{a}_0$, and g_{m2} **is the reflection in the** plane π_m $(m = 0, \pm 1, \pm 2, \ldots)$.

We consider the load $p_{jr}(x, u, t)$ $(j, r = 1, 2)$ that consists of lumped inpulsive forces applied at the points $x = g_{mn}x_{01}$ $(x_{01} = u; \; m = 0, \pm 1, \pm 2, \ldots; \; n = 1, 2)$ and transformed for fixed r into a two-dimensional irreducible representation of the group C_{1h}

$$p_{jr}(x_{01}, u, t) = \delta_{jr}\delta(x_{01} - u)\delta(t) \quad (j, r = 1, 2) \tag{2.3}$$

and the functions $p_{jr}^{(N)}(x, u, t)$ that are finite segments of $p_{jr}(x, u, t)$

$$p_{jr}^{(N)}(x, u, t) = p_{jr}(x, u, t) \quad (x \in X_{mn}; \; |m| \leqslant N; \; n = 1, 2)$$

$$p_{jr}^{(N)}(x, u, t) = 0 \quad (x \in X_{mn}; \; |m| > N; \; n = 1, 2)$$

We find the double transform (Laplace in t and Fourier in x) of the functions $p_{jr}^{(N)}(x, u, t)$:

$$\bar{\bar{p}}_{jr}^{(N)}(\xi, u, s) = \frac{1}{\sqrt{2\pi}} \sum_{q=1}^{2} \sum_{m=-N}^{N} \tau_{jr}^{\mathbf{k}}(g_{mq}) \exp(i2ml\xi - (-1)^q iu\xi)$$

Here and henceforth the double bar denotes double transforms (Laplace and Fourier) and the single bar the Laplace transform.

Since $\tau_{jr}^{\mathbf{k}}(g) = \pm\cos m\alpha, \pm\sin m\alpha$, then for fixed u and s the functions $\bar{\bar{p}}_{jr}^{(N)}(\xi, u, s)$ can be considered to belong to the space of generalized functions of slow growth S' /6/. Passing to the limit as $N \to \infty$ in this space, we have

$$\bar{\bar{p}}(\xi, u, s) = \lim_{N\to\infty} \bar{\bar{p}}_{jr}^{(N)}(\xi, u, s) = (-1)^{j-r} i^{j-1} \times \tag{2.4}$$

$$\frac{T}{\sqrt{2\pi}}\sum_{q=1}^{2}\sum_{k=-\infty}^{\infty}(-1)^{(r-1)(q-1)}\delta(\xi-T(k-(-1)^q\alpha_1))\,z_r(u\xi)$$
$$T=\pi/l,\ \ \alpha_1=\alpha/(2\pi),\ \ z_1(x)=\cos x,\ \ z_2(x)=\sin x$$

Let $v_{jr}^{(N)}(x, t)$ and $v_{jr}(x, t)$ denote the vertical displacements of points of the half-plane boundary due to the action of loads $p_{jr}^{(N)}(x, u, t)$ and $p_{jr}(x, u, t)$, respectively. We use the formula /7/

$$\bar v(x,s)=\frac{s^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sqrt{\xi^2+\beta^2s^2}}{R(\xi,s)}\bar p(\xi,s)\,e^{-ix\xi}\,d\xi \tag{2.5}$$

$$R(\xi,s)=(2\xi^2+s^2)^2-4\xi^2\sqrt{\xi^2+\beta^2s^2}\sqrt{\xi^2+s^2},\quad \beta=\frac{c_2}{c_1}$$

Here $v(x, t)$ is the dimensionless vertical displacement of the boundary points of the half-plane $(v(x, t)=v'(x, t)/a$, where $v'(x, t)$ is the dimensional displacement), $p(x, t)$ is the dimensionless distributed load $(p(x, t)=p'(x, t)/\mu$, μ is the Lamé constant, and $p'(x, t)$ is the dimensional load), c_1 and c_2 are the longitudinal and transverse elastic wave velocities. To extract single-valued branches of the radicals $\sqrt{\xi^2+\beta^2s^2}$ and $\sqrt{\xi^2+s^2}$ for a fixed real ξ in the s plane, slits are made from the branch points $s=\pm i\xi/\beta, \pm i\xi$ to infinity along the appropriate imaginary semi-axis. Those branches of the radicals are selected which take positive values on the real axis of the s-plane.

We continue the functional (2.4) from the space S /6/ to a subspace of functions from $C(-\infty, \infty)$ that tend to zero as $|\xi|\to\infty$. Using (2.5) we have

$$\bar v_{jr}(x,u,s)=\lim_{N\to\infty}\bar v_{jr}^{(N)}(x,u,s)=(-1)^{j-r}\frac{T}{\pi}\sum_{k=-\infty}^{\infty}R_1(\xi_k,s)\,z_j(x\xi_k)\,z_r(u\xi_k)$$
$$\xi_k=T(k+\alpha_1),\quad R_1(\xi,s)=s^2\sqrt{\xi^2+\beta^2s^2}/R(\xi,s)$$

The functions $\bar v_{jr}(x, u, s)$ $(j, r=1, 2)$ enable us to write the transform of the vertical displacements of the boundary points of the half-plane due to the action of arbitrary loads $\bar p_q(x, s)$ $(q=1, 2)$ which transform into an irreducible representation of a symmetry group, as

$$\bar v_j(x,s)=\int_0^l\sum_{q=1}^{2}\bar v_{jr}(x,u,s)\,\bar p_{q01}(u,s)\,du\quad (j=1,2) \tag{2.6}$$
$$\bar p_{qmn}(x,s)=\bar p_q(x,s)\quad (x\in X_{mn})$$

Let P_{j01}^* and M_{j01}^* $(j=1, 2)$ denote the principal vector and principal moment of the base reaction forces acting on the main stamp (located on the basic UC)

$$P_{j01}^*(t)=\int_{d-1}^{d+1}p_{j01}(x_{01},t)\,dx_{01},\quad M_{j01}^*(t)=\int_{d-1}^{d+1}(x_{01}-d)\,p_{j01}(x_{01},t)\,dx_{01} \tag{2.7}$$
$$(P_{jmn}(t)=P'_{jmn}(t)/(a\mu),\quad M_{jmn}(t)=M'_{jmn}(t)/(a^2\mu))$$

Here $p_{j01}(x_{01}, t)$ $(j=1, 2)$ are the contact stresses under the basic stamp under the effect of the loads $P_{jmn}(t)$ and $M_{jmn}(t)$ $(j=1, 2;\ m=0, \pm1, \pm2, \ldots)$, that transform into the irreducible represenation of the group C_{1h} on the system of stamps. The lumped forces $P_{jmn}(t)$ and moments $M_{jmn}(t)$ are dimensionless. Their expressions in terms of the appropriate dimensional quantities are indicated in parentheses.

We write the equation of motion of the basic stamp

$$m_0\frac{d^2w_{j01}^{(1)}}{dt^2}=P_{j01}(t)-P_{j01}^*(t),\quad m_0=m_0'c_2^2/(a^2\mu)$$
$$J_0\frac{d^2\varphi_{01}}{dt^2}=M_{j01}(t)-M_{j01}^*(t),\quad J_0=J_0'c_2^2/(a^4\mu)\quad (j=1,2) \tag{2.8}$$

Here m_0' and J_0' are the dimensional mass and axial moment of inertia of the stamp, $w_{j01}^{(1)}$ and φ_{01} are its dimensionless translational and rotational displacements. Applying the Laplace transform to (2.7) and (2.8), taking into account that the stamps are at rest and have zero initial velocities at the intial time, and assuming small angular displacements, we will have for the displacements of points of the stamp base

$$\bar w_{j01}(x_{01},s)=\frac{\bar P_{j01}(s)-\bar P_{j01}^*(s)}{m_0s^2}+\frac{\bar M_{j01}(s)-\bar M_{j01}^*(s)}{J_0s^2}(x_{01}-d)\quad (j=1,2) \tag{2.9}$$

Equating the right-hand sides of (2.6) and (2.9) for $x_{01} \in [d-1, d+1]$ and making the change of variables $u = \eta + d$, $x_{01} = \xi + d$, we arrive at the system of integral equations

$$\int_{-1}^{1} \sum_{q=1}^{2} \bar{v}_{jq}(\xi + d, \eta + d, s)\, \bar{p}_{q01}(\eta + d, s)\, d\eta = \frac{\bar{P}_{j01}(s) - \bar{P}_{j01}^{*}(s)}{m_0 s^2} + \frac{\bar{M}_{j01}(s) - \bar{M}_{j01}^{*}(s)}{J_0 s^2}\xi \quad (\xi \in [0, 1];\ j = 1, 2) \tag{2.10}$$

3. We seek the solutions of system (2.10) in the form of a series of Chebyshev polynomials

$$\bar{p}_{j01}(\eta + d, s) = \sum_{r=0}^{\infty} \bar{A}_{qr}(s) \frac{T_r(\eta)}{\sqrt{1 - \eta^2}} \tag{3.1}$$

which corresponds to determining the contact stresses by means of the formula

$$p_{j01}(\eta + d, t) = \sum_{r=0}^{\infty} A_{qr}(t) \frac{T_r(\eta)}{\sqrt{1 - \eta^2}}$$

where $A_{qr}(t)$ are unknown functions of time.

We substitute (3.1) into (2.10) and (2.7) for $x_{01} = \xi + d$, then we multiply (2.10) by $T_n(\xi)/\sqrt{1 - \xi^2}$ and we integrate over ξ between -1 and +1. Using the equality ($J_n(x)$ is the Bessel function of the first kind)

$$\int_{-1}^{1} \frac{T_0(x)}{\sqrt{1 - x^2}} dx = \pi, \quad \int_{-1}^{1} \frac{T_{2n}(x)}{\sqrt{1 - x^2}} \cos bx\, dx = (-1)^n \pi J_{2n}(b) \quad (n = 0, 1, 2, \ldots)$$
$$\int_{-1}^{1} \frac{T_{2n+1}(x)}{\sqrt{1 - x^2}} \sin bx\, dx = (-1)^n \pi J_{2n+1}(b) \quad (n = 0, 1, 2, \ldots) \tag{3.2}$$

we will have

$$\bar{P}_{j01}^{*}(s) = \pi \bar{A}_{j0}(s), \quad \bar{M}_{j01}^{*}(s) = \frac{\pi}{2} \bar{A}_{j1}(s) \quad (j = 1, 2) \tag{3.3}$$

$$\sum_{q=1}^{2} \sum_{r=0}^{\infty} B_{jqnr}(s) \bar{A}_{qr}(s) = \delta_{n0} d_0 \frac{\bar{P}_{j01}(s)}{\pi s^2} + \delta_{n1} d_1 \frac{\bar{M}_{j01}(s)}{\pi s^2} \quad (j = 1, 2;\ n = 0, 1, 2, \ldots) \tag{3.4}$$

Here

$$B_{jqnr}(s) = \frac{g_{jqnr}}{s^2} + (-1)^{j+q} \sum_{k=-\infty}^{\infty} \bar{R}_1(\xi_k, s) h_{jkn} h_{qkr} \tag{3.5}$$
$$(j, q = 1, 2;\ n, r = 0, 1, 2, \ldots)$$
$$h_{1kr} = \pi [\delta_r'' (-1)^{r/2} \cos(d\xi_k) - \delta_r' (-1)^{(r-1)/2} \operatorname{sign}(\xi_k) \sin(d\xi_k)] J_r(|\xi_k|)$$
$$h_{2kr} = \pi [\delta_r'' (-1)^{r/2} \sin(d\xi_k) + \delta_r' (-1)^{(r-1)/2} \operatorname{sign}(\xi_k) \cos(d\xi_k)] J_r(|\xi_k|)$$
$$\delta_r' = -\frac{1}{2}[(-1)^r - 1], \quad \delta_r'' = \frac{1}{2}[(-1)^r + 1]$$
$$g_{1100} = g_{2200} = d_0, \quad g_{1111} = g_{2211} = d_1/2, \quad d_0 = \frac{\pi^3}{TM_0}, \quad d_1 = \frac{\pi^3}{2TJ_0}$$

while $g_{jqnr} = 0$ for the remaining values of j, q, n, r.

We make the change of complex variable $s = 1/z$ in the infinite system of linear Eqs. (3.4). Under this transformation, the half-plane $\operatorname{Re} s > 1/2$ is mapped into the interior of the unit circle $|z - 1| < 1$. For a fixed real ξ the function $\bar{R}_1(\xi, s)$ is analytic in the right half-plane $\operatorname{Re} s > 0$ and $\bar{R}_1(\xi, s) \sim \beta/s$ for $\operatorname{Re} s > \gamma > 0$. It hence follows that $\bar{R}_1(\xi_k, 1/z)$ $(k = 0, \pm 1, \pm 2, \ldots)$ is analytic in the circle $|z - 1| < 1$

$$\bar{R}_1\left(\xi_k, \frac{1}{z}\right) = z \sum_{m=0}^{\infty} R_{km} (z - 1)^m$$

Then

$$B_{jqnr}\left(\frac{1}{z}\right) = z \sum_{m=0}^{\infty} B_{jqnrm} (z - 1)^m$$

$$B_{jqnrm} = g_{rqnj} \frac{\delta_{m0} + \delta_{m1}}{2} + (-1)^{j+q} \sum_{k=-\infty}^{\infty} R_{km} h_{jkn} h_{qkr} \tag{3.6}$$
$$(j, q = 1, 2;\ n, r, m = 0, 1, 2, \ldots)$$

In practice, in all the cases encountered in solving the problem, the functions $\bar{P}_{j01}(1/z)$ and $\bar{M}_{j01}(1/z)$ $(j = 1, 2)$ are analytic in the circle $|z-1| < 1$ and are presented there in the form of the series

$$\bar{P}_{j01}\left(\frac{1}{z}\right) = \sum_{m=0}^{\infty} P_{j01m}(z-1)^m, \quad \bar{M}_{j01}\left(\frac{1}{z}\right) = \sum_{m=0}^{\infty} M_{j01m}(z-1)^m \tag{3.7}$$

We will seek the functions $A_{qr}(1/z)$ in the form of the following series

$$A_{qr}\left(\frac{1}{z}\right) = z\sum_{m=0}^{\infty} A_{qrm}(z-1)^m \quad (q = 1, 2;\ r = 0, 1, 2, \ldots) \tag{3.8}$$

Substituting (3.6)-(3.8) into system (3.4) in which we have put $s = 1/z$, and equating coefficients of identical powers of $z-1$, we arrive at a recursion formula for systems of linear algebraic equations in A_{qrm} $(q = 1, 2;\ r, m = 0, 1, 2, \ldots)$

$$\begin{aligned}
&B_0 A_m = D_m \ (m = 0, 1, 2, \ldots) \\
&D_0 = C_0,\ D_m = C_m - \sum_{k=0}^{m-1} B_{m-k} A_k \quad (m = 1, 2, 3, \ldots) \\
&B_m = \| B_{jqnrm} \| \quad (j, q = 1, 2;\ n, r = 0, 1, 2, \ldots) \\
&C_m = \| C_{jnm} \| \quad (j = 1, 2;\ n = 0, 1, 2, \ldots), \\
&A_m = \| A_{qrm} \| \quad (q = 1, 2;\ r = 0, 1, 2, \ldots) \\
&C_{j0m} = d_0 P_{j01m} \quad (j = 1, 2),\quad C_{j1m} = d_1 M_{j01m} \quad (j = 1, 2), \\
&C_{jnm} = 0 \quad (j = 1, 2;\ n \geqslant 2) \quad (m = 0, 1, 2, \ldots)
\end{aligned} \tag{3.9}$$

The convergence of the series on the right-hand side of (3.6) is proved in the same way as the investigation of series (4.4) in /1/. Exactly as in /1/, it is proved that for $n, r \geqslant 1$

$$B_{jjn0} = \frac{C}{n} + O\left(\frac{1}{n^2}\right), \quad |B_{jqnr0}| \leqslant \frac{C}{n^2 r^2} \quad (j \neq q \quad \text{or} \quad n \neq r)$$

where C is a certain constant.

Substituting $A_{qrm} = A_{qrm}^{(1)} / \sqrt{B_{qqrr0}}$ into system (3.9) and dividing the left and right-hand sides of the jn-th row by $\sqrt{B_{jjnn0}}$, we arrive at the system

$$B_0^{(1)} A_m^{(1)} = D_m^{(1)} \tag{3.10}$$

where $B_{jjnn0}^{(1)} = 1$, while for $j \neq q$ or $n \neq r$

$$|B_{jqnr0}^{(1)}| \leqslant \frac{C}{r^{3/2}}\ (n = 0, r \geqslant 1),\ \frac{C}{n^{3/2}}\ (r = 0, n \geqslant 1),\ \frac{C}{(nr)^{3/2}}\ (n, r \geqslant 1) \tag{3.11}$$

It follows from inequalities (3.11) that (3.10) is a normal system /8/, which means that it, and system (3.9), are solved by using reduction.

Acting as in /1/, it can be seen that the coefficients of the series (3.8) satisfy the asymptotic form $A_{qrm} = O(1/m^2)$.

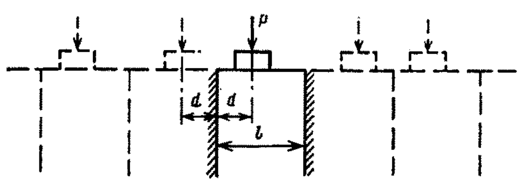

Fig.3

As a result of the successive solution of the abbreviated systems for $m = 0, 1, 2, \ldots$ we obtain the coefficients A_{qrm} $(q = 1, 2;\ r = 0, 1, 2, \ldots, N;\ m = 0, 1, 2, \ldots)$ of the expansion (3.8). Making the inverse substitution $z = 1/s$ and going over to the originals, we obtain ($L_m(t)$ is the Laguerre polynomial)

$$A_{qr}(t) = \sum_{m=0}^{\infty} A_{qrm}(-1)^m L_m(t) \quad (q = 1, 2;\ r = 0, 1, 2, \ldots, N)$$

The generalized symmetric problem is solved in exactly the same way when the appropriate irreducible representation of the group C_{1h} is one-dimensional. In this case, the functions of the basis (1.1) are transformed into a one-dimensional real representation. Carrying out calculations similar to those presented above, we reduce the problem to an integral equation of the form (2.10).

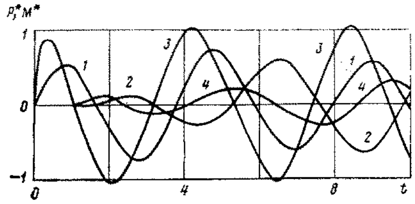

Fig.4

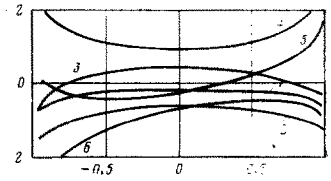

Fig.5

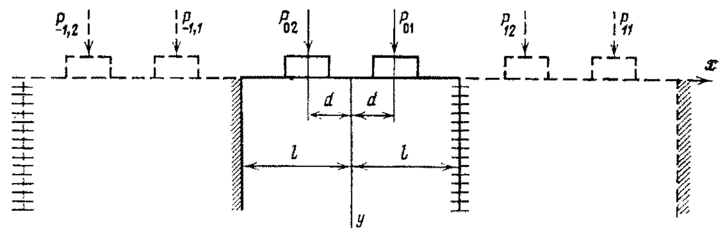

Fig.6

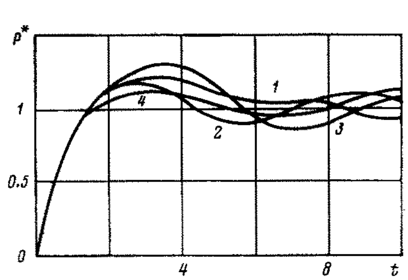

Fig.7

The contact stresses determined in the solution of the generalized symmetric problems should be summed in order to evaluate the contact stresses under the effect of an arbitrary load from Δ_X on the stamp.

Example 1. A stamp at whose middle a vertical harmonic force $P(t) = H(t)\cos(\omega t)$ is applied, where $H(t)$ is the Heaviside unit function (Fig.3), is at the endface of an elastic half-strip between two ideally smooth absolutely stiff guides. This problem is equivalent to a periodic contact problem with stamps arranged in pairs. In this case the load is converted into the one-dimensional representation with $\alpha = 0$. As a result of applying the algorithm formulated above, we determine the contact stresses for different values of t. Graphs of the change in the principal vector P^* and the principal moment M^* of the reactive forces of the base in the time interval $[0, 10]$ are presented in Fig.4 for $\omega = 1.5$, $l = 4$, $d = 1.5$. Curves *1* and *2* correspond to the change in P^* and M^* for $d_0 = 12.5$, $d_1 = 10$; curves *3* and *4* for $d_0 = 100$ and $d_1 = 50$. Contact stress diagrams are presented in Fig.5 for $\omega = 1.5$, $l = 4$, $d = 1.5$, $d_0 = 12.5$, $d_1 = 10$ for the times $t = 0.2; 0.8; 1.8; 2.8; 3.8; 4.8$ (curves *1-6*, respectively).

Example 2. Two stamps are arranged symmetrically at the endface of the half-strip presented in Fig.6. The left face of the half-strip abuts against an ideally smooth absolutely stiff surface (the "sliding" face), and constraints hindering the vertical displacements of points of the faces (the "support" face) are at the right side. At $t = 0$ vertical unit forces $P = H(t)$ are suddenly applied to the middle of the stamps. The problem of determining the contact stresses under these stamps is equivalent to the contact problem for a half-plane with a periodically duplicated system of two stamps. The half-plane is bonded from the half-strips mentioned above (Fig.6). The load is continued from the given (basic) half-strip to the rest so that it consequently turns out to be symmetric with respect to the sliding faces and anti-symmetric with respect to the support faces. It can be seen that $P_{mn}(t) = -\sqrt{2}H(t)\sin[(m-1)\pi/2]$. In this case a generalized symmetric problem corresponding to a two-dimensional irreducible representation of the group C_{1h} with the parameter $\alpha = \pi/2$ must be solved. Graphs of the changes in the reaction under the stamps located on the basic half-strip are presented in Fig.7 as a function of t. Curves *1* and *2* correspond to the reactions under the right and left stamps for $d=2$, curves *3* and *4* for $d=2.5$ and the values of the remaining parameters are $l=4, d_0=12.5, d_1=10$.

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Translated by M.D.F.

PMM U.S.S.R.,Vol.50,No.3,pp.357-361,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

ON THE NON-LINEAR BOUNDARY EQUATIONS OF THE MECHANICS OF THE CONTACT OF ROUGH ELASTIC BODIES*

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As a generalization of the results of /1/, the existence and uniqueness of the solution of the contact problem of several rough elastic bodies (in the example of the contact of three bodies is proved). The method of non-linear boundary equations is used, which (like the variational method /2-5/) enables an effective investigation to be made of the correctness of the problem of body contact with unknown contact domains.

1. Formulation of the problem. Let $Oxyz$ be a Cartesian rectangular coordinate system, $M(x, y)$ a point in the plane $E_2=\{z=0\}$ with the coordinates x, y, $\operatorname{mes}\{\omega\}$ the Lebesgue measure of the set $\omega \subset E_2$; Δ^2 the biharmonic operator, $\operatorname{supp} v(M)$ a carrier of the function $v(M)$, Q an operator setting the function $v(M)$, $M \in \Omega$ in correspondence with the function $v^+(M)$, $M \in \Omega$ according to the rule $v^+ \equiv Qv = \sup\{v(M), 0\}$, and $L_{r,2} = L_{r,2}(\Omega)$ a Banch space of the vector-functions $v(M) = (v_1(M), v_2(M))$ (defined in the domain $\Omega \subset E_2$) with the norm

$$\|v(M)\| = \Big(\int_\Omega (|v_1(M)|^r + |v_2(M)|^r dS_M\Big)^{1/r}, \quad r \geqslant 1$$

For $r=2$ the space $L_{r,2}$ is a Hilbert space with the scalar product

$$(u, v) = \int_\Omega (u_1(M)v_1(M) + u_2(M)v_2(M))\,dS_M$$

The linear operator K acting from the Banach space E on the conjugate space E^* to E is called strictly positive if $(Kv, v) \geqslant 0$, and the equality $(Kv, v)=0$ is possible if and only if $v=0$ /6/ ((u, v) is the value of the linear continuous functional $u \in E^*$ at the element $v \in E$).

We will consider the (frictionless) contact problem of an elastic body *1* and elastic half-space *3* with a plate *2* located between them (see the sketch, the y axis is perpendicular to the plane of the sketch). As a boundary value problem it reduces (with known assumptions) to the construction of the respective harmonic functions $u_1(x, y, z)$, $u_3(x, y, z)$ in the half-spaces

*Prikl.Matem.Mekhan.,50,3,470-474,1986